Functional Analysis

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Lecture 6

Hilbert spaces

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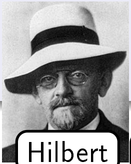
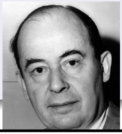
Inner product

Def. Inner product space (or **pre-Hilbert space**) is a linear space H over $\mathbb{F} = \mathbb{R}, \mathbb{C}$ equipped with an **inner product**, that is a function $\langle \cdot, \cdot \rangle : H \times H \to \mathbb{F}$ satisfying

1. $\langle x, x \rangle \geqslant 0$ and $\langle x, x \rangle > 0$ for $x \neq 0$ (*positive-definiteness*)
2. $\langle \alpha x + \beta y, z \rangle = \alpha \langle x, z \rangle + \beta \langle y, z \rangle$, (*linearity in the first argument*)
3. $\langle x, y \rangle = \overline{\langle y, x \rangle}$, (*antisymmetry*)

for x, y, $z \in H$ and $\alpha, \beta \in \mathbb{F}$.


Hilbert


von Neumann

Rem. Conditions (2) and (3), imply

$$\langle x, \alpha y + \beta z \rangle = \overline{\alpha} \langle x, y \rangle + \overline{\beta} \langle x, z \rangle$$ (*antilinearity in the second argument*)

Hence the inner product is a **sesquilinear form!**

If $\mathbb{F} = \mathbb{R}$, then (3) assume the form $\langle x, y \rangle = \langle y, x \rangle$. Hence **a real inner product** $\equiv$ **a bilinear symmetric form**!

Ex1. $\langle x, y\rangle = \sum_{k=1}^{n} x(k)\overline{y(k)}$ defines an inner product on $H = \mathbb{F}^n$.

Ex2. $\langle x, y\rangle = \sum_{k=1}^{\infty} x(k)\overline{y(k)}$ defines an inner product on the space of square summable sequences $H = \ell^2$.

Ex3. $\langle x, y\rangle = \int_\Omega x(t)\cdot\overline{y(t)}\,d\mu$ defines an inner product on the space of square integrable functions $H = L^2(\mu)$.

Rem. Hölder's inequality for $p = q = 2$, that is

$$\int_\Omega |x(t)y(t)|\,d\mu \leqslant \|x\|_2 \cdot \|y\|_2,$$

is often called **Schwartz ineguality**. It guaranties that the inner product on is well-defined : $x, y \in L^2(\mu) \Rightarrow x\cdot\overline{y} \in L^1(\mu)$. Notice that

$$\|x\|_2 = \left(\int_\Omega |x(t)|^2\,d\mu\right)^{\frac{1}{2}} = \sqrt{\langle x, x\rangle}.$$

Def. The **norm** of $x \in H$ in an inner product space H is given by

$$\|x\| := \sqrt{\langle x, x\rangle}.$$

Lem. $\underbrace{\|x+y\|^2 = \|x\|^2 + 2\,\mathrm{Re}\langle x, y\rangle + \|y\|^2}_{\text{“shortened multiplication formula”}}$ for $x, y \in H$

$(a+b)^2 = a^2 + 2ab + b^2$

Thm. (abstract Schwartz inequality)

For any x, y in an inner product space we have

$$|\langle x, y\rangle| \leqslant \|x\| \cdot \|y\|.$$

Proof: Let $\lambda \in \mathbb{F}$ be such that $|\langle x, y\rangle| = \lambda \cdot \langle x, y\rangle$ and $|\lambda| = 1$ ($\lambda = e^{-i \arg\langle x,y\rangle}$). For any $t \in \mathbb{R}$ we have

$$0 \leqslant \langle t\lambda x + y, t\lambda x + y\rangle = \|t\lambda x\|^2 + 2\,\mathrm{Re}\langle t\lambda x, y\rangle + \|y\|^2$$
$$= t^2\|x\|^2 + 2t|\langle x, y\rangle| + \|y\|^2.$$

Hence the function $f(t) := t^2\|x\|^2 + 2t|\langle x, y\rangle| + \|y\|^2$ is postive.

Thus the discriminant is negative: $\Delta = 4|\langle x, y\rangle|^2 - 4\|x\|^2\|y\|^2 \leqslant 0$.

After transformations this gives $|\langle x, y\rangle| \leqslant \|x\| \cdot \|y\|$. ∎

Cor. The function $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$ is a norm on an inner product space.

Proof: $\|x\| = 0 \iff \langle x, x\rangle = 0 \iff x = 0$. For $\lambda \in \mathbb{F}$ we have

$$\|\lambda x\| = \sqrt{\langle \lambda x, \lambda x\rangle} = \sqrt{\lambda\bar{\lambda}\langle x, x\rangle} = \sqrt{|\lambda|^2\langle x, x\rangle} = |\lambda| \cdot \|x\|.$$

For $x, y \in H$ we have

$$\|x+y\|^2 = \|x\|^2 + 2\operatorname{Re}\langle x, y\rangle + \|y\|^2 \overset{\operatorname{Re} z \leqslant |z|}{\leqslant} \|x\|^2 + 2|\langle x, y\rangle| + \|y\|^2$$
$$\overset{Schwartz}{\leqslant} \|x\|^2 + 2\|x\| \cdot \|y\| + \|y\|^2 = (\|x\| + \|y\|)^2.$$

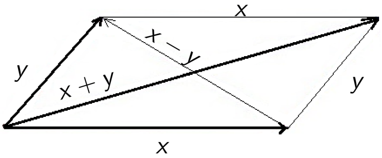
Taking the square root we get the triangle inequality ■

Def. Hilbert space = inner product space, which is complete in the norm $\|\cdot\| = \sqrt{\langle\cdot,\cdot\rangle}$.

Rem. The inner product $\langle\cdot,\cdot\rangle : H \times H \to \mathbb{F}$ is continuous in the norm (this follows from the Schwartz inequality). In particular, it extends to the completion of H. Therefore,

the complemetion of an inner product space is a Hilbert space!

The parallelogram law states that: *the sum of the squares of the lengths of the sides is equal to the sum of the squares of the lengths of the diagonals*



Theorem.

In every inner product space the **parallelogram law** holds:

$$\forall_{x,y\in H} \qquad \|x+y\|^2 + \|x-y\|^2 = 2\left(\|x\|^2 + \|y\|^2\right)$$

and the inner product is determined via the **polarization identity**:

$$\langle x, y\rangle = \begin{cases} \frac{1}{4}(\|x+y\|^2 - \|x-y\|^2), & \text{when } \mathbb{F} = \mathbb{R}, \\ \frac{1}{4}\sum_{k=0}^{3} i^k \|x + i^k y\|^2, & \text{when } \mathbb{F} = \mathbb{C}. \end{cases}$$

Conversely, if $(H, \|\cdot\|)$ is a normed space in which the parallelogram law holds, then the polarization formula gives an inner product on H such that $\|x\| = \sqrt{\langle x, x\rangle}$, so H is an inner product space.

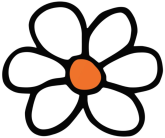
Dowód: '$\Longrightarrow$' '$\Longleftarrow$' for the ambitious.

Cor. A Banach space $(X, \|\cdot\|)$ is a Hilbert space if and only if the norm $\|\cdot\|$ satisfies the parallelogram law.

Prop. $L^p(\mu)$, $p \in [1, \infty)$, is a Hilbert space if and only if $p = 2$.

Proof: We $L^2(\mu)$ is a Hilbert space, where $\langle x, y\rangle = \int x \cdot \overline{y}\, d\mu$. Assume that parallegram law holds in $L^p(\mu)$. Take any disjoint A, B with $0 < \mu(A), \mu(B) < \infty$. Putting $x := \mathbb{1}_A$ and $y = \mathbb{1}_B$ we have

$$\|x + y\|_p^2 + \|x - y\|_p^2 = 2\mu(A \cup B)^{\frac{2}{p}},$$
$$2\left(\|x\|_p^2 + \|y\|_p^2\right) = 2\left(\mu(A)^{\frac{2}{p}} + \mu(B)^{\frac{2}{p}}\right).$$



Dividing sides by sides $1 = \left(\frac{\mu(A)}{\mu(A\cup B)}\right)^{\frac{2}{p}} + \left(\frac{\mu(B)}{\mu(A\cup B)}\right)^{\frac{2}{p}}$. Hence putting $\lambda_1 := \frac{\mu(A)}{\mu(A\cup B)}$ and $\lambda_2 := \frac{\mu(B)}{\mu(A\cup B)}$ we get numbers satisfying

$$0 < \lambda_1, \lambda_2 < 1, \qquad \lambda_1 + \lambda_2 = 1 = (\lambda_1)^{\frac{2}{p}} + (\lambda_2)^{\frac{2}{p}}.$$

These relations imply that $p = 2$, because if $p > 2$ then $\lambda_i > (\lambda_i)^{\frac{2}{p}}$, and if $p < 2$, then $\lambda_i < (\lambda_i)^{\frac{2}{p}}$ dla $i = 1, 2$.